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A robust generalized Bayes estimator improving on the James-Stein estimator for spherically symmetric distributions

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Summary: The problem of estimating a mean vector for spherically symmetric distributions with the quadratic loss function is considered. A robust generalized Bayes estimator improving on the James-Stein estimator is given.

1 Introduction

Consider the linear regression model

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{\beta} + \boldsymbol{e} \tag{1.1}$$

where $\boldsymbol{y}$ is an $N \times 1$ response vector, $\boldsymbol{A}$ is an $N \times p$ matrix of rank $p \leq N$ of known constants, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, and $\boldsymbol{e}$ is an $N \times 1$ vector of unobservable random errors. We assume that the error $\boldsymbol{e}$ has a spherically symmetric density $\sigma^{-N} f(\|\boldsymbol{e}\|^2/\sigma^2)$, where σ^2 is an unknown parameter and $f(\cdot)$ is a nonnegative function on the nonnegative real line. We can easily derive the canonical form of (1.1). Let $\boldsymbol{P}$ be an $N \times N$ orthogonal matrix such that

$$\boldsymbol{P}\boldsymbol{A} = \begin{pmatrix} (\boldsymbol{A}'\boldsymbol{A})^{1/2} \\ \boldsymbol{0} \end{pmatrix}$$

and let $\boldsymbol{\theta} = (\boldsymbol{A}'\boldsymbol{A})^{1/2}\boldsymbol{\beta}$. Hence two random vectors $\boldsymbol{X} = (X_1, \cdots, X_p)'$ and $\boldsymbol{Z} = (Z_1, \cdots, Z_n)'$ where

$$\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Z} \end{pmatrix} = \boldsymbol{P}\boldsymbol{Y} \quad \text{and} \quad n = N - p$$

have the joint density of the form of

$$\sigma^{-p-n} f((\|\boldsymbol{x} - \boldsymbol{\theta}\|^2 + \|\boldsymbol{z}\|^2)/\sigma^2), \tag{1.2}$$

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where $\boldsymbol{\theta}$ is a $p \times 1$ vector of unknown parameters. References on these distributions which generalize the multivariate normal distribution in linear regression model are given by Kelker [8], Eaton [5], Fang and Anderson [6] and Kubokawa and Srivastava [11]. Then we consider the problem of estimating the mean vector $\boldsymbol{\theta}$ by $\boldsymbol{\delta}(\boldsymbol{X}, \boldsymbol{Z})$ relative to the quadratic loss function $L(\boldsymbol{\theta}, \sigma^2, \boldsymbol{d}) = \|\boldsymbol{d} - \boldsymbol{\theta}\|^2/\sigma^2$.

For the normal model, it is well-known that the usual minimax estimator $\boldsymbol{X}$ is inadmissible for $p \geq 3$ as shown in Stein [14]. James and Stein [7] succeeded in giving an explicit form of an estimator dominating $\boldsymbol{X}$ as

$$\boldsymbol{\delta}^{JS}(\boldsymbol{X}, \boldsymbol{Z}) = \left(1 - \frac{p-2}{n+2}\frac{\|\boldsymbol{Z}\|^2}{\|\boldsymbol{X}\|^2}\right)\boldsymbol{X},$$

which is called the James-Stein estimator. For the spherically symmetric model, Cellier et al. [4] and Kubokawa and Srivastava [11] showed that the James-Stein estimator dominates the usual estimator $\boldsymbol{X}$ independent of f in (1.2), which does not need to be known. However it turns out that the James-Stein estimator is inadmissible since its positive-part estimator is superior to it as shown in Baranchik [2]. Moreover its positive-part estimator is not analytic and thus inadmissible. Therefore it has been of interest to derive analytic (generalized Bayes, if possible) estimators improving on $\boldsymbol{X}$ and the James-Stein estimator. For the normal model, Lin and Tsai [12] derived a class of minimax generalized Bayes estimators. Kubokawa [9] showed that the estimator $\boldsymbol{\delta}_K = (1 - \phi_K(W)/W)\boldsymbol{X}$, where $W = \|\boldsymbol{X}\|^2/\|\boldsymbol{Z}\|^2$ and

$$\phi_K(w) = w\frac{\int_0^1 \lambda^{p/2-1}(1+\lambda w)^{-(n+p)/2-1}d\lambda}{\int_0^1 \lambda^{p/2-2}(1+\lambda w)^{-(n+p)/2-1}d\lambda}, \tag{1.3}$$

which is included in Lin and Tsai's class, dominates the James-Stein estimator. Moreover Kubokawa [10] derived a sufficient condition for domination over the James-Stein estimator and showed that $\boldsymbol{\delta}_K$ satisfies it. Maruyama [13] showed that some generalized Bayes estimators besides $\boldsymbol{\delta}_K$ satisfy Kubokawa's [10] condition. However such estimators have not been derived for the spherically symmetric model yet.

In this paper, we show that Lin and Tsai [12] and Kubokawa's [9, 10] results remain robust under a broad subclass of spherically symmetric distributions although these seem to depend upon the normality. In particular we recommend the use of $\boldsymbol{\delta}_K$ for any spherical symmetric distribution since it is minimax for any such f, dominates the James-Stein estimator for those f which are unimodal, and is also generalized Bayes under the condition of a finite fourth moment.

2 Generalized Bayes estimators for spherically symmetric distributions

Letting $\eta = \sigma^{-2}$, we consider the prior distribution whose joint density of $\boldsymbol{\theta}$ and η is proportional to $\eta^a\|\boldsymbol{\theta}\|^{-b}$. From the fact

$$\|\boldsymbol{\theta}\|^{-b} = \frac{\eta^{b/2}}{\Gamma(b/2)2^{b/2}}\int_0^1 \lambda^{b/2-1}(1-\lambda)^{-b/2-1}\exp\left(-\frac{\eta\lambda}{2(1-\lambda)}\|\boldsymbol{\theta}\|^2\right)d\lambda, \tag{2.1}$$

for $b > 0$, this prior is interpreted as the hierarchical prior

$$\boldsymbol{\theta}|\lambda,\eta \sim N_p\left(\mathbf{0}, \eta^{-1}\frac{1-\lambda}{\lambda}I_p\right), \quad \lambda \propto \lambda^{b/2-p/2-1}(1-\lambda)^{-b/2+p/2-1}, \quad \eta \propto \eta^{b/2-p/2+a},$$

which is a special case of ones considered in Lin and Tsai [12] and Alam [1] in the normal case of our problem. In the estimation of a multivariate normal mean with known variance, Baranchik [2] investigated the generalized Bayes estimator with respect to $\|\boldsymbol{\theta}\|^{-b}$.
Under the quadratic loss function $\eta\|\boldsymbol{d}-\boldsymbol{\theta}\|^2$, the generalized Bayes estimator is given by $E(\eta\boldsymbol{\theta}|\boldsymbol{X},\boldsymbol{Z})/E(\eta|\boldsymbol{X},\boldsymbol{Z})$ and we have the generalized Bayes estimator with respect to our prior,

$$\begin{aligned}
&\frac{\int_{R^p}\int_0^\infty \boldsymbol{\theta}\eta^{(n+p)/2+a+1}f(\eta(\|\boldsymbol{X}-\boldsymbol{\theta}\|^2+\|\boldsymbol{Z}\|^2))\|\boldsymbol{\theta}\|^{-b}d\boldsymbol{\theta}\,d\eta}{\int_{R^p}\int_0^\infty \eta^{(n+p)/2+a+1}f(\eta(\|\boldsymbol{X}-\boldsymbol{\theta}\|^2+\|\boldsymbol{Z}\|^2))\|\boldsymbol{\theta}\|^{-b}d\boldsymbol{\theta}\,d\eta}\\
&\quad= \frac{\int_{R^p}\boldsymbol{\theta}(\|\boldsymbol{X}-\boldsymbol{\theta}\|^2+\|\boldsymbol{Z}\|^2)^{-(p+n)/2-a-2}\|\boldsymbol{\theta}\|^{-b}d\boldsymbol{\theta}\cdot\int_0^\infty \eta^{(n+p)/2+a+1}f(\eta)d\eta}{\int_{R^p}(\|\boldsymbol{X}-\boldsymbol{\theta}\|^2+\|\boldsymbol{Z}\|^2)^{-(p+n)/2-a-2}\|\boldsymbol{\theta}\|^{-b}d\boldsymbol{\theta}\cdot\int_0^\infty \eta^{(n+p)/2+a+1}f(\eta)d\eta}\\
&\quad= \frac{\int_{R^p}\int_0^\infty \boldsymbol{\theta}\tau^{(p+n)/2+a+1}\exp(-\tau(\|\boldsymbol{X}-\boldsymbol{\theta}\|^2+\|\boldsymbol{Z}\|^2)/2)\|\boldsymbol{\theta}\|^{-b}d\boldsymbol{\theta}\,d\tau}{\int_{R^p}\int_0^\infty \tau^{(p+n)/2+a+1}\exp(-\tau(\|\boldsymbol{X}-\boldsymbol{\theta}\|^2+\|\boldsymbol{Z}\|^2)/2)\|\boldsymbol{\theta}\|^{-b}d\boldsymbol{\theta}\,d\tau}
\end{aligned}$$

if $\int \eta^{(n+p)/2+a+1}f(\eta)d\eta < \infty$, which is equivalent to the finiteness of the $2(a+2)$-th moment of the distribution of $\boldsymbol{X}$ and $\boldsymbol{Z}$. That is to say, the generalized Bayes estimator under the spherically symmetric case does not depend on f and hence coincides with one under the normal case. From (2.1) we have

$$\begin{aligned}
\int_{R^p}\boldsymbol{\theta}\exp\left(-\frac{\tau}{2}\|\boldsymbol{x}-\boldsymbol{\theta}\|^2-\frac{\tau\lambda\|\boldsymbol{\theta}\|^2}{2(1-\lambda)}\right)d\boldsymbol{\theta} &= \left(\frac{2\pi(1-\lambda)}{\tau}\right)^{p/2}\exp(-\lambda\tau\|\boldsymbol{x}\|^2/2)(1-\lambda)\boldsymbol{x}\\
\int_{R^p}\exp\left(-\frac{\tau}{2}\|\boldsymbol{x}-\boldsymbol{\theta}\|^2-\frac{\tau\lambda\|\boldsymbol{\theta}\|^2}{2(1-\lambda)}\right)d\boldsymbol{\theta} &= \left(\frac{2\pi(1-\lambda)}{\tau}\right)^{p/2}\exp(-\lambda\tau\|\boldsymbol{x}\|^2/2).
\end{aligned}$$

Moreover from the relation

$$\int_0^\infty \tau^{n/2+a+1+b/2}\exp\left(-\frac{\tau(\lambda\|\boldsymbol{x}\|^2+\|\boldsymbol{z}\|^2)}{2}\right) = \frac{\Gamma(n/2+a+b/2+2)2^{n/2+a+b/2+2}}{(\|\boldsymbol{z}\|^2(1+w\lambda))^{n/2+a+b/2+2}}$$

where $w = \|\boldsymbol{x}\|^2/\|\boldsymbol{z}\|^2$, we have the generalized Bayes estimator $\boldsymbol{\delta}_{a,b}(\boldsymbol{X},\boldsymbol{Z}) = (1-\phi_{a,b}(W)/W)\boldsymbol{X}$ where

$$\phi_{a,b}(w) = w\frac{\int_0^1 \lambda^{b/2}(1-\lambda)^{p/2-b/2-1}(1+w\lambda)^{-n/2-a-b/2-2}d\lambda}{\int_0^1 \lambda^{b/2-1}(1-\lambda)^{p/2-b/2-1}(1+w\lambda)^{-n/2-a-b/2-2}d\lambda}, \tag{2.2}$$

which is well-defined if $0 < b < p$ and $n/2+a+b/2+2 > 0$. Note that $\boldsymbol{\delta}_K = \boldsymbol{\delta}_{a,b}$ for $a=0$ and $b=p-2$.
Here we summarize the result on the property of $\boldsymbol{\delta}_{a,b}$.

Theorem 2.1 *For $0 < b < p$, $n/2 + a + b/2 + 2 > 0$ and any spherically symmetric distribution, the $2(a+2)$-th moment of which is finite, $\boldsymbol{\delta}_{a,b}$ is generalized Bayes with respect to the density $\eta^a\|\boldsymbol{\theta}\|^{-b}$.*

The properties of the behavior of $\phi_{a,b}(w)$ is as follows.

Theorem 2.2 *1. $\phi_{a,b}(w)$ is monotone increasing in w if $0 < p \leq p-2$.*

2. $\lim_{w\to\infty} \phi_{a,b}(w) = b/(n+2a+2)$ if $a > -n/2-1$ and

$$|\phi_{a,b}(w) - b/(n+2a+2)| = \begin{cases} O\{(w+1)^{-n/2-a-1}\} & \text{for } \ b = p-2 \\ O\{(w+1)^{-1}\} & \text{for } \ 0 < b < p-2. \end{cases}$$

Proof: By the change of variables, we have

$$\phi_{a,b}(w) = \frac{\int_0^w t^{b/2}(1-t/w)^{p/2-b/2-1}(1+t)^{-n/2-a-b/2-2}dt}{\int_0^w t^{b/2-1}(1-t/w)^{p/2-b/2-1}(1+t)^{-n/2-a-b/2-2}dt}.$$

For $w_1 > w_2$ and $0 < b \leq p-2$,

$$\begin{aligned} &\frac{\int_0^{w_1} t^{b/2}(w_1-t)^{p/2-b/2-1}(1+t)^{-n/2-a-b/2-2}dt}{\int_0^{w_1} t^{b/2-1}(w_1-t)^{p/2-b/2-1}(1+t)^{-n/2-a-b/2-2}dt} \\ &\geq \frac{\int_0^{w_2} t^{b/2}(w_1-t)^{p/2-b/2-1}(1+t)^{-n/2-a-b/2-2}dt}{\int_0^{w_2} t^{b/2-1}(w_1-t)^{p/2-b/2-1}(1+t)^{-n/2-a-b/2-2}dt} \\ &\geq \frac{\int_0^{w_2} t^{b/2}(w_2-t)^{p/2-b/2-1}(1+t)^{-n/2-a-b/2-2}dt}{\int_0^{w_2} t^{b/2-1}(w_2-t)^{p/2-b/2-1}(1+t)^{-n/2-a-b/2-2}dt}. \end{aligned}$$

The first inequality is from the fact that the ratio of integrands of the numerator and the denominator is increasing, the second inequality from the fact that $\{(w_2-t)/(w_1-t)\}^{p/2-b/2-1}$ is increasing. This completes the proof of part 1.
¿From an identity

$$\int_0^1 \lambda^\alpha(1-\lambda)^\beta(1+w\lambda)^{-\gamma}d\lambda = (w+1)^{-\alpha-1}\int_0^1 t^\alpha(1-t)^\beta(1-tw/(w+1)^{-\alpha-\beta+\gamma-2}dt,$$

we have

$$\phi_{a,b}(w) = v\frac{\int_0^1 t^{b/2}(1-t)^{p/2-b/2-1}(1-vt)^{-p/2+n/2+a+b/2+1}dt}{\int_0^1 t^{b/2-1}(1-t)^{p/2-b/2-1}(1-vt)^{-p/2+n/2+a+b/2+2}dt}$$

for $v = w/(w+1)$, which implies that

$$\lim_{w\to\infty}\phi_{a,b}(w) = \frac{\int_0^1 t^{b/2}(1-t)^{n/2+a}dt}{\int_0^1 t^{b/2-1}(1-t)^{n/2+a+1}dt} = \frac{b}{n+2a+2},$$

for $0 < b < p$ and $a > -n/2 - 1$. When $0 < b \le p - 2$, applying an integration by parts gives

$$\phi_{a,b}(w) = -\frac{(n/2+a+1)^{-1}(1-v)^{n/2+a+1}}{\int_0^1 t^{p/2-2}(1-vt)^{n/2+a+1}dt} + \frac{b}{n+2a+2} \tag{2.3}$$

for $b = p - 2$ and

$$\begin{aligned}\phi_{a,b}(w) = &\frac{b}{n+2a+2} - \frac{p-b-2}{n+2a+2}(1-v) \\ &\times \frac{\int_0^1 t^{b/2}(1-t)^{p/2-b/2-2}(1-vt)^{-p/2+n/2+a+b/2+1}dt}{\int_0^1 t^{b/2-1}(1-t)^{p/2-b/2-1}(1-vt)^{-p/2+n/2+a+b/2+2}dt},\end{aligned} \tag{2.4}$$

for $0 < b < p - 2$. From (2.3) and (2.4), we can easily see that $|b/(n+2a+2) - \phi_{a,b}| = O\{(w+1)^{-n/2-a-1}\}$ for $b = p - 2$ and $= O\{(w+1)^{-1}\}$ for $0 < b < p - 2$, which completes the proof of part 2. □

3 Improving on $\boldsymbol{X}$ and the James-Stein estimator

For the spherically symmetric model, Cellier *et al.* [4] and Kubokawa and Srivastava [11] gave the generalized Stein identity and chi-square identity

$$E\left[(X_i - \theta_i)h(\boldsymbol{X})\right] = \sigma^2 \iint \frac{\partial}{\partial x_i} h(\boldsymbol{x}) F\left(\frac{\|\boldsymbol{x}-\boldsymbol{\theta}\|^2 + \|\boldsymbol{z}\|^2}{\sigma^2}\right) d\boldsymbol{x}\, d\boldsymbol{z},$$

$$E\left[\|\boldsymbol{z}\|^2 g(\|\boldsymbol{z}\|^2)\right] = \sigma^2 \iint \{ng(\|\boldsymbol{z}\|^2) + 2\|\boldsymbol{z}\|^2 g'(\|\boldsymbol{z}\|^2)\} F\left(\frac{\|\boldsymbol{x}-\boldsymbol{\theta}\|^2 + \|\boldsymbol{z}\|^2}{\sigma^2}\right) d\boldsymbol{x}\, d\boldsymbol{z},$$

for suitable f and g. By using these identities, the risk of an estimator of the form

$$\boldsymbol{\delta}_\phi(\boldsymbol{X}, \boldsymbol{Z}) = (1 - \phi(\|\boldsymbol{X}\|^2/\|\boldsymbol{Z}\|^2)\|\boldsymbol{Z}\|^2/\|\boldsymbol{X}\|^2)\boldsymbol{X}$$

is expressed as

$$\begin{aligned} R(\boldsymbol{\theta}, \sigma^2, \boldsymbol{\delta}_\phi) = &E\left[\frac{\|\boldsymbol{X}-\boldsymbol{\theta}\|^2}{\sigma^2}\right] + \sigma^{-2}E\left[\phi^2\left(\frac{\|\boldsymbol{X}\|^2}{\|\boldsymbol{Z}\|^2}\right)\frac{\|\boldsymbol{Z}\|^4}{\|\boldsymbol{X}\|^2}\right] \\ &+ 2E\left[\frac{(\boldsymbol{X}-\boldsymbol{\theta})'\boldsymbol{X}}{\sigma^2}\phi\left(\frac{\|\boldsymbol{X}\|^2}{\|\boldsymbol{Z}\|^2}\right)\frac{\|\boldsymbol{Z}\|^2}{\|\boldsymbol{X}\|^2}\right] \\ = &p + \int_{R^p}\int_{R^n}\bigg(\frac{\phi(w)}{w}\{(n+2)\phi(w) - 2(p-2)\} \\ &- 4\phi'(w)(1+\phi(w))\bigg)\sigma^{-p-n}F((\|\boldsymbol{x}-\boldsymbol{\theta}\|^2 + \|\boldsymbol{z}\|^2)/\sigma^2)d\boldsymbol{x}\, d\boldsymbol{z}, \end{aligned} \tag{3.1}$$

where $w = \|\boldsymbol{x}\|^2/\|\boldsymbol{z}\|^2$ and $F(u) = 2^{-1}\int_u^\infty f(t)dt$. Hence a sufficient condition for dominance over the natural estimator $\boldsymbol{X}$ is derived as follows.

Theorem 3.1 (Kubokawa and Srivastava) *Assume that $\phi(w)$ is monotone nondecreasing and $0 \le \phi(w) \le 2(p-2)/(n+2)$ for every $w \ge 0$. Then $\boldsymbol{\delta}_\phi$ dominates $\boldsymbol{X}$.*

This sufficient condition under the normal case was derived by Baranchik [3].
Among a simple class of shrinkage estimators $\boldsymbol{\delta}_\phi$ with $\phi(w) = c$, the optimal c, denoted by c^*, is $(p-2)/(n+2)$ and $\boldsymbol{\delta}_{c^*}$ is of course the James-Stein estimator. It is well-known that the James-Stein estimator is inadmissible since its positive-part estimator dominates it. In the following, we present a sufficient condition for dominance over the James-Stein estimator which is the generalization of Kubokawa's [10] result.

Theorem 3.2 *Assume that $\phi(w)$ is monotone nondecreasing, $\lim_{w\to\infty}\phi(w) = (p-2)/(n+2)$ and $\phi(w) \geq \phi_K(w)$ for every $w \geq 0$. Then $\boldsymbol{\delta}_\phi$ dominates the James-Stein estimator for unimodal spherically symmetric distributions.*

Proof: By letting $\Phi(w) = w^{-1}\phi\{(n+2)\phi - 2(p-2)\} - 4\phi'(1+\phi)$ and by using the transformation to the polar coordinates, the second term of the right-hand side of (3.1) is written as

$$\begin{aligned}
&\int_{R^p}\int_{R^n} \Phi(w)\sigma^{-p-n}F((\|\boldsymbol{x}-\boldsymbol{\theta}\|^2 + \|\boldsymbol{z}\|^2)/\sigma^2)d\boldsymbol{x}\,d\boldsymbol{z} \\
&= \int_{R^p}\int_{R^n} \Phi\left(\frac{\|\boldsymbol{x}\|^2}{\|\boldsymbol{z}\|^2}\right) F(\|\boldsymbol{x}-\boldsymbol{\theta}/\sigma\|^2 + \|\boldsymbol{z}\|^2)d\boldsymbol{x}\,d\boldsymbol{z} \\
&= C\int_0^\infty\int_0^\infty\int_0^\pi \Phi\left(\frac{s^2}{t^2}\right) F(s^2 - 2s\lambda^{1/2}\cos\varphi + \lambda + t^2) \\
&\quad \cdot \sin^{p-2}\varphi s^{p-1}t^{n-1}ds\,dt\,d\varphi \\
&= \frac{C}{2}\int_0^\infty\int_0^\infty\int_0^\pi \Phi(w)F(s^2 - 2s\lambda^{1/2}\cos\varphi + \lambda + s^2/w) \\
&\quad \cdot w^{-(n+1)/2}\sin^{p-2}\varphi s^{p+n-1}dsdwd\varphi \\
&= \frac{C}{2}\int_0^\infty\int_0^\infty\int_0^\pi \Phi(w)F(u^2 - 2\lambda^{1/2}(1+1/w)^{-1/2}u\cos\varphi + \lambda) \\
&\quad \cdot w^{(p-2)/2}(1+w)^{-(p+n)/2}u^{p+n-1}\sin^{p-2}\varphi\,du\,dw\,d\varphi,
\end{aligned}$$

where $C = 4\pi^{(p+n-1)/2}/\{\Gamma((p-1)/2)\Gamma(n/2)\}$ and $\lambda = \|\boldsymbol{\theta}\|^2/\sigma^2$. Letting

$$\begin{aligned}
g_\lambda(w) = &\frac{w^{p/2-1}}{(1+w)^{(p+n)/2}}\int_0^\pi\int_0^\infty F(u^2 - 2\lambda^{1/2}(1+1/w)^{-1/2}u\cos\varphi + \lambda) \\
&\cdot u^{p+n-1}\sin^{p-2}\varphi\,d\varphi\,du,
\end{aligned}$$

we have

$$R(\boldsymbol{\theta},\sigma^2,\boldsymbol{\delta}^{JS}) - R(\boldsymbol{\theta},\sigma^2,\boldsymbol{\delta}_\phi) = \frac{C}{2}\int_0^\infty \left(-\frac{(p-2)^2}{(n+2)w} - \Phi(w)\right) g_\lambda(w)dw.$$

By a definite integral

$$\left[v(w)\int_0^w s^{-1}g_\lambda(s)ds\right]_0^\infty = \int_0^\infty v'(w)\int_0^w s^{-1}g_\lambda(s)dsdw + \int_0^\infty \frac{v(w)}{w}g_\lambda(w)dw$$

for a differentiable function $v(w)$. Letting $(w) = \phi\{(n+2)\phi - 2(p-2)\}$ in the above equality and noting that $\lim_{w\to\infty}\phi(w) = (p-2)/(n+2)$, we have

$$\begin{aligned} &R(\boldsymbol{\theta},\sigma^2,\boldsymbol{\delta}^{JS}) - R(\boldsymbol{\theta},\sigma^2,\boldsymbol{\delta}_\phi) \\ &= C\int_0^\infty \phi'(w)\left[\{(n+2)\phi(w)-p+2\}\int_0^w s^{-1}g_\lambda(s)ds + 2(1+\phi(w))g_\lambda(w)\right]dw. \end{aligned} \tag{3.2}$$

Since $g_\lambda(w)/g_0(w)$ is nondecreasing in w by Lemma 3.3 in the below, we have

$$g_\lambda(w)/\int_0^w s^{-1}g_\lambda(s)ds \geq g_0(w)/\int_0^w s^{-1}g_0(s)ds,$$

and hence we have

$$\begin{aligned} &R(\boldsymbol{\theta},\sigma^2,\boldsymbol{\delta}^{JS}) - R(\boldsymbol{\theta},\sigma^2,\boldsymbol{\delta}_\phi) \\ &\geq C\int_0^\infty \phi'(w)\left[(n+2)\phi(w)-p+2+2\frac{(1+\phi(w))g_0(w)}{\int_0^w s^{-1}g_0(s)ds}\right]\int_0^w s^{-1}g_\lambda(s)ds\,dw. \end{aligned}$$

Since $\phi_K(w)$ can be expressed as

$$\phi_K(w) = \frac{(p-2)\int_0^w s^{-1}g_0(s)ds - 2g_0(w)}{(n+2)\int_0^w s^{-1}g_0(s)ds + 2g_0(w)} \tag{3.3}$$

we have the theorem. □

Combining (3.2) and (3.3), we have $R(\mathbf{0},\sigma^2,\boldsymbol{\delta}^{JS}) = R(\mathbf{0},\sigma^2,\boldsymbol{\delta}_K)$, which implies that $\boldsymbol{\delta}_K$ can never dominate the James-Stein positive-part estimator unfortunately.

Lemma 3.3 *$g_\lambda(w)/g_0(w)$ is nondecreasing in w if $f(\cdot)$ is monotone nonincreasing.*

We note that the unimodality assumption corresponds to the fact the function f is nonincreasing.

Proof: We have only to show that

$$h(w) = \int_0^\pi F(u^2 - 2\lambda^{1/2}(1+1/w)^{-1/2}u\cos\varphi + \lambda)\sin^{p-2}\varphi d\varphi,$$

for fixed u and λ is nondecreasing. The derivative of $h(w)$ is

$$\begin{aligned} \frac{d}{dw}h(w) =& d\int_0^\pi f(u^2 - 2\lambda^{1/2}(1+1/w)^{-1/2}u\cos\varphi + \lambda)\cos\varphi\sin^{p-2}\varphi d\varphi \\ =& d\int_0^{\pi/2} f(u^2 - 2\lambda^{1/2}(1+1/w)^{-1/2}u\cos\varphi + \lambda)\cos\varphi\sin^{p-2}\varphi d\varphi \\ &- d\int_0^{\pi/2} f(u^2 + 2\lambda^{1/2}(1+1/w)^{-1/2}u\cos\varphi + \lambda)\cos\varphi\sin^{p-2}\varphi d\varphi \end{aligned}$$

where $d = 2^{-1}w^{-2}(1+1/w)^{-3/2}\lambda^{1/2}u$. The assumption of the lemma on f guarantees that $h(w)$ is nondecreasing in w. $\square$

Combining Theorem 3.1 and 3.2, we have the results on the decision-theoretic properties of $\boldsymbol{\delta}_{a,b}$.

Theorem 3.4 *1.* $\boldsymbol{\delta}_{a,b}$ *is minimax for any spherically symmetric distributions if* $a > -n/2-1$, $0 < b \le p-2$ *and* $b/(n+2a+2) < 2(p-2)/(n+2)$.

2. $\boldsymbol{\delta}_K$*, which equals to* $\boldsymbol{\delta}_{0,p-2}$*, dominates the James-Stein estimator under the unimodal spherically symmetric distributions.*

Note that among the minimax generalized Bayes estimators $\boldsymbol{\delta}_{a,b}$, only $\boldsymbol{\delta}_{0,p-2}$ satisfies the sufficient condition of Theorem 3.2. We thus recommend the use of $\boldsymbol{\delta}_K$ for any spherical symmetric distribution since it is minimax for any such f, dominates the James-Stein estimator for those f which are unimodal, and is also generalized Bayes under the condition of a finite fourth moment.

Remark 3.5 The problem addressed in the paper is expected to have a relationship with some other estimation problems, in particular, the estimation of a scale parameter σ^2 with an unknown $\boldsymbol{\theta}$ and the estimation of $\boldsymbol{\theta}$ with known σ^2. But unfortunately we cannot establish a similar robustness property.

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